Admissible sets for polytopic linear systems subject to slowly-varying unobservable disturbances

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***Abstract*— This paper describes a method to calculate sets of safe initial conditions for polytopic linear systems subject to slowly-varying unobservable disturbances. Such sets are called admissible sets; invariant sets are admissible but admissible sets are not necessarily invariant. By examining conditions for admissible sets of this particular system class, we prove that admissible sets can be calculated by only considering a special property that we call *the λ-proximal-contraction property* at the vertices of the set that bounds the slowly-varying disturbances. An academic example is used to illustrate how to apply the method, and a discussion follows that summarizes the main points.**

I. Introduction

Set invariance has been a research topic within the control community for at least five decades [1]. It can be used to analyze safety properties of dynamical systems and to synthesize robust controllers [2], [3], [4], [5]. The techniques are attractive for safety critical applications, but their worst-case nature can lead to conservative restrictions on the applications' operational domains.

Provided a dynamical-system description of a safety application, it is natural to search for the largest set of initial states whose state trajectories never violate safety constraints. If no additional information is provided for parameters and disturbances than their bounds, then the Maximal Robust Positive Invariant (MRPI) set is the largest set that respects the safety-critical constraints (see Definition 5). However, in many practical problems for safety-critical systems the derived system description provides more information about parameters and disturbances than only their bounds.

To exemplify, external signals that act on a control system may have *a priori* known properties which limit their rate of change. In applications for autonomous cars, examples of this include (but are not limited to) self-aligning wheel torques and road-banking angles. Even if the signal can be measured, the sensor may not provide high enough integrity from a safety-critical perspective to consider the signal observable. Robust invariance cannot directly take *a priori* information such as bounded rates of change into consideration. And to model the signal as a state-dependent disturbance, as in [6], is not very useful if it is not observable.

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The insight that the MRPI set may not always be the most suitable representation of an admissible set is the main motivation for analyzing admissible sets for slowly-varying systems in [7]. Slowness is there captured in the model by including a bound on the rate of change of a scalar parameter, and the authors state an algorithm that produces admissible sets that are supersets of the corresponding MRPI sets. One drawback of that algorithm, however, lies in its computational complexity when applied to higher-dimensional slowly-varying parameters. In this paper, we derive properties of polytopic linear systems with slowly-varying disturbances that enables an alternative way to calculate admissible sets, that can be significantly less demanding than applying the algorithm from [7].

II. Preliminaries

A. Notation

The symbol $\mathbb{R}$ denotes the set of reals; $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$ are the integers, non-negative integers and positive integers, respectively; $\mathbb{R}^n$ is the set of real-valued n-vectors, and $\mathbb{R}^{m\times n}$ is the set of real $m \times n$ matrices; $\mathcal{I}_{i_0,\dots,i_n}$ denotes the index set $\left\{(i_j+\ell)_{j=0,\ell\in\{0,1\}}^{n}\right\}$. We denote matrices with capital letters (e.g., A, B), vectors and scalars with lowercase letters (e.g., x, w, p), elements of vectors by the use of subscripts (e.g., x_i), and sets by calligraphic letters (e.g., $\mathcal{X}$, $\mathcal{W}$, $\mathcal{C}$). Logical expressions are written in prenex normal form when quantifiers are used.

B. System with slowly-varying disturbances

We herein focus on systems of the form

$$x(k+1) = Ax(k) + Bw(k) + Dp(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $w(k) \in \mathcal{W} \subseteq \mathbb{R}^m$, and $D \in \mathbb{R}^{n\times\ell}$. To start off, we let $\ell = 1$. Without loss of generality we assume $p \in \mathcal{P} = [0,1]$ and that its rate of change is known to be bounded by $\forall k \in \mathbb{Z}\ |p(k+1) - p(k)| \le \epsilon$. For the sake of a clearer presentation, we assume that $1/\epsilon$ is an integer, and for the purpose of calculating admissible sets we divide the parameter interval into N overlapping subintervals,

$$\mathcal{P}^i = [p^i, p^{i+2}] \subseteq \mathcal{P}, \tag{2a}$$
$$p^{i+1} - p^i = \epsilon, \quad p^0 = 0, \quad p^{N+1} = 1, \tag{2b}$$

for $i = 0,\dots,N-1$. The partition is illustrated in Figure 1. The subintervals are ordered and their union equals the parameter interval.

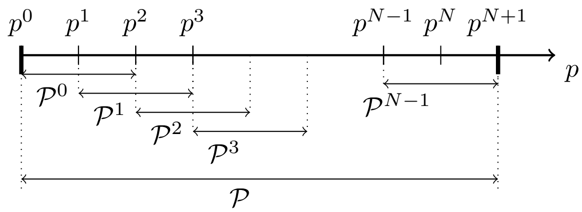

Fig. 1. Illustration of the intervals and notations introduced in (2). Each subinterval has length 2ϵ, each overlap has length ϵ, and the union of the subintervals is identical to the whole parameter interval.

C. Useful definitions

Definition 1 (Constraint set): A set that outlines safety-critical constraints in the state space is said to be a *constraint set*, and we denote this by $\mathcal{X}$.

Definition 2 (One-step robust backward-reachable set): For system (1), the *one-step robust backward-reachable set* to the set $\mathcal{S}$ is defined as $\mathbf{Pre}(\mathcal{S}, \mathcal{W}, \mathcal{P}) = \{x \in \mathbb{R}^n : \forall w \in \mathcal{W},\ \forall p \in \mathcal{P}, Ax + Bw + Dp \in \mathcal{S}\}$.
For brevity we will write $\mathbf{Pre}_i(\mathcal{S})$ as shorthand for $\mathbf{Pre}(\mathcal{S}, \mathcal{W}, \mathcal{P}^i)$.

Definition 3 (k-step robust backward-reachable set): For a given set $\mathcal{S}$, we define the *k-step robust backward-reachable set* $\mathcal{K}_k(\mathcal{S}, \mathcal{W}, \mathcal{P})$ of system (1) as the recursion

$$\mathcal{K}_j(\mathcal{S}, \mathcal{W}, \mathcal{P}) = \mathbf{Pre}(\mathcal{K}_{j-1}(\mathcal{S}, \mathcal{W}, \mathcal{P}), \mathcal{W}, \mathcal{P}),$$
$$\mathcal{K}_0(\mathcal{S}, \mathcal{W}, \mathcal{P}) = \mathcal{S}, \quad j = 1, \ldots, k.$$

This way to define the k-step robust backward-reachable set differs from how it is defined in for example [8], while it is aligned with the treatment of controllability in [1].

Definition 4 (RPI set): A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be *robust positive invariant* (RPI) for system (1) if $x(k) \in \mathcal{O} \Rightarrow x(k+1) \in \mathcal{O}$.
If only subinterval $\mathcal{P}^i$ is considered, we denote the corresponding set by $\mathcal{O}^i$.

Definition 5 (MRPI set): A set $\mathcal{O}_\infty \subseteq \mathcal{X}$ is said to be *maximal robust positive invariant* (MRPI) for system (1) if it is RPI and contains all RPI sets.
If only subinterval $\mathcal{P}^i$ is considered, we denote the corresponding MRPI set by $\mathcal{O}^i_\infty$.

Definition 6 (λ-contractive RPI set): We say that the set $\mathcal{S}$ is a *λ-contractive robust positive invariant set* for system (1) if $x(k) \in \mathcal{S} \Rightarrow x(k+1) \in \lambda\mathcal{S}$ for some $\lambda \in [0, 1]$.

Definition 7 (λ-MRPI set): A set $\mathcal{C}_\lambda$ is is said to be *maximal λ-contractive robust positive invariant* (λ-MRPI) for system (1) if it is λ-contractive RPI and contains all λ-contractive RPI sets for the system.
If only subinterval $\mathcal{P}^i$ is considered, we denote the corresponding λ-MRPI set by $\mathcal{C}^i_\lambda$.

Definition 8 (Safe initial condition): We say that an initial condition is a *safe initial condition* if $x(0) \in \mathcal{X}$ implies $x(k) \in \mathcal{X}$ for all $k \in \mathbb{Z}_+$.

Definition 9 (Admissible set): A set of safe initial conditions is called an *admissible set*.

Definition 10 (λ-proximal-contraction property): We say that the λ-MRPI set $\mathcal{C}^i_\lambda$ satisfies the *λ-proximal-contraction property* if $\lambda\mathcal{C}^i_\lambda \subseteq \bigcap_{j=i-1}^{i+1} \mathcal{C}^j_\lambda$.

In other words, a λ-MRPI set $\mathcal{C}^i_\lambda$ that satisfies the λ-proximal-contraction property ensures that if $x(k) \in \mathcal{C}^i_\lambda$ and $p(k) \in \mathcal{P}^i$, then $x(k+1) \in \mathcal{C}^{i-1}_\lambda$ and $x(k+1) \in \mathcal{C}^{i+1}_\lambda$.

D. A conceptual algorithm

One way way to find the MRPI set for a system on the form (1) is to apply Algorithm 1.

Algorithm 1 Calculation of MRPI set.

Input: g, $\mathcal{X}$, $\mathcal{W}$, $\mathcal{P}$
Here, g is a system on the form (1), and the constraint set $\mathcal{X}$ is compact.
Output: $\mathcal{O}_\infty$ (the MRPI set)

let $\Omega_0 \leftarrow \mathcal{X}$, $k \leftarrow 0$
repeat
  $\Omega_{k+1} \leftarrow \mathbf{Pre}(\Omega_k, \mathcal{W}, \mathcal{P}) \cap \mathcal{X}$
  $k \leftarrow k + 1$
until $\Omega_k \supseteq \Omega_{k-1}$
$\mathcal{O}_\infty \leftarrow \Omega_k$
return $\mathcal{O}_\infty$

Variants of this algorithm is often referred to as a principal way to calculate MRPI sets, and it is also implemented in software such as the Multi-Parametric Toolbox (MPT) [9]. It should be remarked for later that this algorithm produces the sequence of sets $\Omega_k = \bigcap_{j=0}^k \mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P})$.

III. Admissible sets for the system

With the goal to calculate admissible sets, we now go on to derive properties of system (1) that, step by step, takes us to the main result of this contribution.

A. Some properties of the system

Lemma 1: A set $\mathcal{S}_i$ is λ-MRPI for system (1) (for a particular subinterval $\mathcal{P}^i$) if and only if it is MRPI for $x(k+1) = \lambda^{-1}\big(Ax(k) + Bw(k) + Dp\big)$.

Proof: See Lemma 4.28 and Lemma 4.29 in [1]. ■

Lemma 2: If $\mathcal{C}^i_\lambda$, $i = 0, \ldots, N-1$, are nonempty and satisfy the λ-proximal-contraction property, then an admissible set for system (1) is $\bigcap_{i=0}^{N-1} \mathcal{C}^i_\lambda$.

Proof: If $x(0) \in \bigcap_{i=0}^{N-1} \mathcal{C}^i_\lambda$, then for any initial value $p(0) \in \mathcal{P}^i$ for the parameter, $x(0)$ lies in the invariant set $\mathcal{C}^i_\lambda$. For all future values of k, $x(k)$ will lie in $\mathcal{C}^{j(k)}_\lambda$ when $p(k) \in \mathcal{P}^{j(k)}$, $j(k) \in \{0, \ldots, N-1\}$, due to the λ-proximal-contraction property. Hence, the state trajectory cannot leave $\mathcal{X}$, and therefore respects the safety constraints. ■

Lemma 3: Let $\mathcal{X}$ be a polyhedron. Then the hyperplanes that define Ω^i_k when using Algorithm 1 lie between the corresponding hyperplanes for Ω^0_k and Ω^{N-1}_k.

Proof: If $\mathcal{H} = \{x \in \mathbb{R}^n : Hx \leq h\}$ represents a halfspace in $\mathcal{X}$, then the corresponding halfspaces generated by the algorithm for interval $\mathcal{P}^i$ can be represented as

$$\mathcal{K}_k = \Big\{x \in \mathbb{R}^n : HA^k x \leq h - H\textstyle\sum_{j=0}^{k-1} A^j B w^*_j - H\sum_{j=0}^{k-1} A^j D p^*_j - p^i H \sum_{j=0}^{k-1} A^j D\Big\}, \tag{3}$$

where $\{w_j^*\}$ is the sequence of worst-case disturbances that moves the hyperplane towards the origin, and $\{p_j^*\}$ is the sequence of worst-case parameter values that moves the hyperplane towards the origin within the interval $\mathcal{P}^0$. The extreme values for the right hand side of the inequality (i.e., that determines the distance from the origin to the hyperplane) occur for $i = 0$ and $i = N-1$. Hence the result follows. ∎

Corollary 1: Let n_i^k denote the shortest (normal) vector from the origin to the hyperplane that defines the halfspace $\mathcal{K}_k(\mathcal{H}, \mathcal{W}, \mathcal{P}^i)$. Then,

$$n_i^k = \frac{i}{N-1} n_{N-1}^k + \left(1 - \frac{i}{N-1}\right) n_0^k,$$

which in turn implies $\left\|n_{i+1}^k - n_i^k\right\| = \left\|n_1^k - n_0^k\right\|$ for $i = 0, \dots, N-2$.

Proof: This is seen by inspecting Equation (3). ∎

The property stated in Corollary 1 is illustrated in Figure 2.

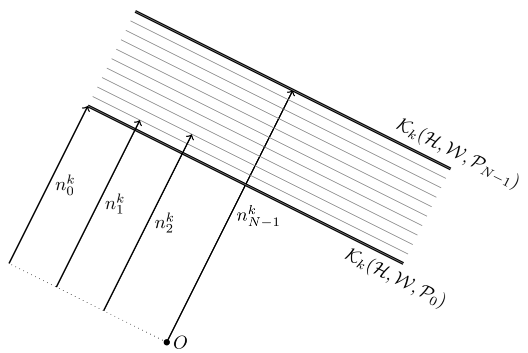

Fig. 2. Corollary 1 says that the distance between n_{i+1}^k and n_i^k is the same for $i = 0, \dots, N-2$. Either n_0^k or n_{N-1}^k lies closest to the origin for the general case.

Lemma 4: Assume for some $\lambda \in [0, 1]$ that $\mathcal{C}_\lambda^0$ and $\mathcal{C}_\lambda^{N-1}$ are nonempty. Then, $\mathcal{C}_\lambda^i$ is nonempty with $\mathcal{C}_\lambda^0 \cap \mathcal{C}_\lambda^{N-1} \subseteq \mathcal{C}_\lambda^i$, for $i = 0, \dots, N-1$.

Proof: By Lemma 1, the λ-MRPI set, $\mathcal{C}_\lambda^i$, is the MRPI set for the system $x(k+1) = \frac{1}{\lambda}\big(Ax(k) + Bw(k) + pD\big)$, $p \in \mathcal{P}^i$, and it holds for this system that $\mathcal{C}_\lambda^i = \bigcap_{k=0}^\infty \mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P}^i)$. By Lemma 3, the hyperplanes that define $\mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P}^i)$ lie between the corresponding hyperplanes that define $\mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P}^0)$ and $\mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P}^{N-1})$. Therefore, $\mathcal{C}_\lambda^0 \cap \mathcal{C}_\lambda^{N-1} = \bigcap_{k=0}^\infty \big(\mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P}^0) \cap \mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P}^{N-1})\big) \subseteq \mathcal{C}_\lambda^i = \bigcap_{k=0}^\infty \mathcal{K}_k(\mathcal{X}, \mathcal{W}, \mathcal{P}^i)$. ∎

Corollary 2: With the assumptions stated in Lemma 4 it holds that $\mathcal{C}_\lambda^0 \cap \mathcal{C}_\lambda^{N-1} \subseteq \bigcap_{i=0}^{N-1} \mathcal{C}_\lambda^i$.

Lemma 5: That $\mathcal{C}_\lambda^i$ for some $i \in \{1, \dots, N-2\}$ satisfies the λ-proximal-contractive property is equivalent to $\left\|n_1^k - n_0^k\right\| \le (1-\lambda)\left\|n_i^k\right\|$ for all $k \in \mathbb{Z}_{++}$. That $\mathcal{C}_\lambda^0$ and $\mathcal{C}_\lambda^{N-1}$ both satisfy the λ-proximal-contractive property is equivalent to $\left\|n_1^k - n_0^k\right\| \le (1-\lambda)\min(\left\|n_0^k\right\|, \left\|n_{N-1}^k\right\|)$ for all $k \in \mathbb{Z}_{++}$.

Proof: For the first part, the property is equivalent to $\lambda\left\|n_i^k\right\| \le \left\|n_{i\pm1}^k\right\| \Leftrightarrow \left\|n_i^k\right\| - \lambda\left\|n_i^k\right\| \ge \left\|n_i^k\right\| - \left\|n_{i\pm1}^k\right\| \Leftrightarrow (1-\lambda)\left\|n_i^k\right\| \ge \left\|n_{i\pm1}^k - n_i^k\right\| = \left\|n_1^k - n_0^k\right\|$, where the last equivalence step follows since one of $\left\|n_{i\pm1}^k\right\| - \left\|n_i^k\right\|$ is positive and both have the same magnitude according to Corollary 1. The second part can be proved using the same technique while also accounting for which of $\left\|n_0^k\right\|$ and $\left\|n_{N-1}^k\right\|$ has the largest magnitude. ∎

Lemma 6: Assume that $\mathcal{C}_\lambda^0$ and $\mathcal{C}_\lambda^{N-1}$ are nonempty and satisfy the λ-proximal-contraction property. Then, $\mathcal{C}_\lambda^i$ is nonempty for $i = 0, \dots, N-1$, and they all satisfy the λ-proximal-contraction property.

Proof: Existence follows from Lemma 4. Lemma 5 implies $\left\|n_1^k - n_0^k\right\| \le (1-\lambda)\min(\left\|n_0^k\right\|, \left\|n_{N-1}^k\right\|) \le (1-\lambda)\left\|n_i^k\right\|$ for all k, which implies $\mathcal{C}_\lambda^i$ satisfies the λ-proximal-contraction property for $i = 0, \dots, N-1$. ∎

B. Main result

We have now arrived at the main result of this contribution.

Theorem 1: If $\mathcal{C}_\lambda^0$ and $\mathcal{C}_\lambda^{N-1}$ are nonempty and satisfy the λ-proximal-contraction property, then an admissible set for system (1) is the intersection $\mathcal{C}_\lambda^0 \cap \mathcal{C}_\lambda^{N-1}$.

Proof: We need to show that the conditions in Lemma 2 hold. To this end, combine Lemma 6 and Corollary 2. ∎

C. Polytopic linear systems

It is straightforward to verify that all derived results hold for polytopic linear systems; that is, systems of the form $x(k+1) = A(k)x(k) + B(k)w(k) + D(k)p(k)$, where

$$\begin{aligned}
A(k) &\in \mathcal{A} = \{A : \textstyle\sum_{i=1}^\kappa \mu_i A_i,\ \sum_{i=1}^\kappa \mu_i = 1,\ \mu_i \ge 0\}\,;\\
B(k) &\in \mathcal{B} = \{B : \textstyle\sum_{i=1}^\kappa \mu_i B_i,\ \sum_{i=1}^\kappa \mu_i = 1,\ \mu_i \ge 0\}\,;\\
D(k) &\in \mathcal{D} = \{D : \textstyle\sum_{i=1}^\kappa \mu_i D_i,\ \sum_{i=1}^\kappa \mu_i = 1,\ \mu_i \ge 0\}\,.
\end{aligned}$$

D. The multidimensional case

Many practical systems are subject to more than one slowly-varying exogenous disturbance. To extend the results to this case, we consider the system

$$\begin{aligned}
x(k+1) &= Ax(k) + Bw(k) + Dp(k),\\
&= Ax(k) + Bw(k) + \textstyle\sum_{i=1}^\ell D^{(i)} p_i(k),
\end{aligned} \tag{4}$$

where $D = \begin{bmatrix} D^{(1)} & \cdots & D^{(\ell)} \end{bmatrix}$, $p = \begin{bmatrix} p_1 & \cdots & p_\ell \end{bmatrix}^T$, $D^{(i)} \in \mathbb{R}^n$ and $p_i \in \mathcal{P}^{(i)} = [0, 1]$ for $i = 1, \dots, \ell$ are independent scalar parameters subject to $|p_i(k+1) - p_i(k)| \le \epsilon_i$. The parameter is now bounded by a hypercube, $\mathcal{P} = \mathcal{P}^{(1)} \times \cdots \times \mathcal{P}^{(\ell)}$. Each subinterval becomes a subhypercube, as illustrated for the two-dimensional case in Figure 3. For simplicity we still refer to them as subintervals.

The nomenclature for the high-dimensional case augments the scalar case to multi-indexed subscripts for the subintervals and corresponding contractive sets, $\mathcal{P}^{i_1,\dots,i_\ell}$ and $\mathcal{C}_\lambda^{i_1,\dots,i_\ell}$, where i_j range from 0 to $N_j - 1$ for $j = 1, \dots, \ell$. The MRPI sets related to each subinterval are denoted by $\mathcal{O}_\infty^{i_1,\dots,i_\ell}$.

Definition 11 (λ-proximal-contraction property): The set $\mathcal{C}_\lambda^{i_1,\dots,i_\ell}$ is said to satisfy the *λ-proximal-contraction property* if

$$\lambda \mathcal{C}_\lambda^{i_1,\dots,i_\ell} \subseteq \bigcap_{j_1=i_1-1}^{i_1+1} \cdots \bigcap_{j_\ell=i_\ell-1}^{i_\ell+1} \mathcal{C}_\lambda^{j_1,j_2,\dots,j_\ell}. \tag{5}$$

Definition 11 is a generalization of Definition 10. If the state lies inside a set with the λ-proximal-contraction property its

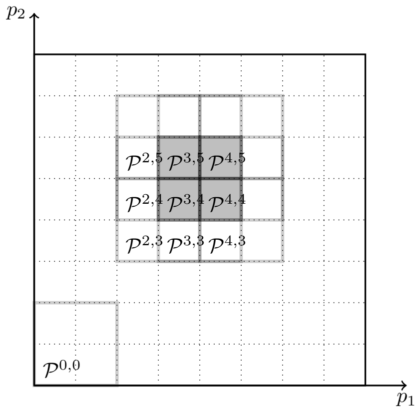

Fig. 3. Subintervals are subhypercubes in higher dimensions, here illustrated in two dimensions. To check if $\mathcal{C}_\lambda^{3,4}$ satisfies the λ-proximal-contraction property one can calculate $\mathcal{C}_\lambda^{i_1,i_2}$ for all neighboring subintervals and then determine if (5) holds.

successor will lie inside the corresponding λ-contractive sets for the neighboring subintervals.

Theorem 1 is straightforward to generalize to the high-dimensional case:

Corollary 3: Assume that $\mathcal{C}_\lambda^{i_1,\dots,i_\ell}$ satisfies the λ-proximal-contraction property for $i_j \in \{0, N-1\}$, for $j = 1,\dots,\ell$. Then, $\bigcap_{i_1\in\{0,N_1-1\}} \cdots \bigcap_{i_\ell\in\{0,N_\ell-1\}} \mathcal{C}_\lambda^{i_1,\dots,i_\ell}$ is an admissible set.
A straightforward proof of this can be made using the same ideas as for the one-dimensional case, but is here omitted in favor of brevity. The usefulness of Corollary 3 is illustrated by a hypothetical example in Figure 4.

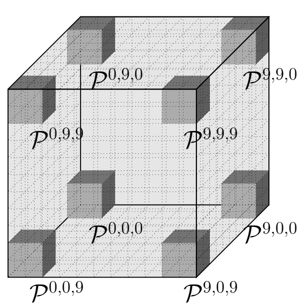

Fig. 4. Illustration of the consequence of Corollary 3. For higher-dimensional disturbances that are slowly varying, it suffices to check the λ-proximal-contraction property for $\mathcal{C}_\lambda^{i_1,i_2,i_3}$ that correspond to the subintervals at the vertices of the set that bounds the slowly-varying disturbance. For the example in the picture this requires 8 such calculations. Without the insights gained by Theorem 1 and Corollary 3 one would require 10^3 such calculations for this example to calculate an admissible set. Problems in higher dimensions and more densely gridded intervals (small ϵ) adds to the usefulness of the result.

IV. Numerical examples

A. Unobservable reference signal

To illustrate how Corollary 3 may be used in practice, consider the problem of finding an admissible set for a simple control system subject to a safety-critical model-following constraint. The system is depicted in Figure 5. The plant is

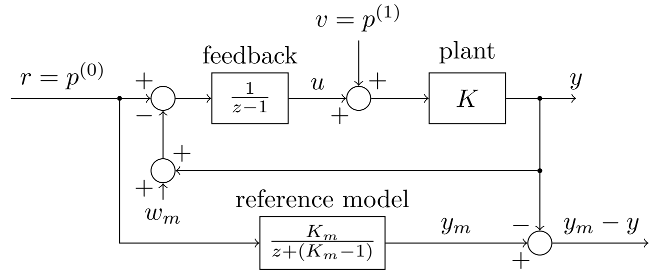

Fig. 5. Control system with model-following requirement. In general, a two-degrees of freedom controller is preferred to separate disturbance attenuation and reference following, but for this example a one-degree of freedom controller makes the admissible set possible to visualize in two dimensions.

modeled as an uncertain constant, $\forall k\ K(k) \in [0.9, 1.1]$, and the reference model uses $K_m = 1$. The reference input, r, and the disturbance, v, cannot be determined beforehand, but it is known that

$$\begin{aligned} \forall k \in \mathbb{Z}, \quad r(k) \in [-1, 1], \quad |r(k+1) - r(k)| \le 0.1, \\ \forall k \in \mathbb{Z}, \quad v(k) \in [-1, 1], \quad |v(k+1) - v(k)| \le 0.1. \end{aligned} \tag{6}$$

The measurement noise, w_m, is subject to $\forall k \in \mathbb{Z}$, $w_m(k) \in [-0.01, 0.01]$. The safety-critical model-following requirement is assumed to be $\forall k \in \mathbb{Z}$, $|y_m(k) - y(k)| \le 0.5$.

Writing the system on the form (4), with $x_1 = y_m$ and $x_2 = y$, yields

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} 1-K_m & 0 \\ 0 & 1-K \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &\quad + \underbrace{\begin{bmatrix} 0 & 0 \\ -K & K \end{bmatrix}}_{B} w(k) + \underbrace{\begin{bmatrix} K_m & 0 \\ K & K-1 \end{bmatrix}}_{D} \begin{bmatrix} p^{(0)} \\ p^{(1)} \end{bmatrix}, \end{aligned}$$

where we introduced $w = \begin{bmatrix} w_m & \tilde{v} \end{bmatrix}^T$ and $\tilde{v}(k) = v(k) - v(k-1)$. It follows that $\mathcal{W} = \{w : |w_1| \le 0.01,\ |w_2| \le 0.1\}$ and $\mathcal{X} = \{x : |x_1 - x_2| \le 0.5\}$. No invariant sets exist for this system.

By knowledge of the bounded rates of change, we select $p^{(0)} = r$ and $p^{(1)} = v$ and divide the interval space into a 20-by-20 grid with 19×19 overlapping subintervals of width 0.2 in each dimension. Interval $\mathcal{P}^{i_1,i_2}$ corresponds to $r \in [-1+0.1i_1, -0.8+0.1i_1]$ and $v \in [-1+0.1i_2, -0.8+0.1i_2]$ for $i_1 = 0,\dots,19$, $i_2 = 0,\dots,19$.

To apply Corollary 3 we need to determine sets for the subintervals at the vertices of $\mathcal{P}$ that satisfy the λ-proximal-contraction property. To that end, we try $\lambda = 0.5$ and find the MRPI set for the system $x(k+1) = \lambda^{-1}\big(Ax(k) + Bw(k) + Dp(k)\big)$, $p \in \mathcal{P}^{i_0,i_1}$ for $(i_0, i_1) \in \mathcal{I}_{0,0} = \{(0,0), (0,1), (1,0), (1,1)\}$. This returns the sets $\mathcal{C}_{i_0,i_1}$ for $(i_0,i_1) \in \mathcal{I}_{0,0}$. For their intersection we then evaluate if $\mathbf{Pre}\left(\bigcap_{(i_0,i_1)\in\mathcal{I}_{0,0}} \mathcal{C}_{i_0,i_1}\right) \in \mathcal{C}_{i_0,i_1}$, $(i_0,i_1) \in \mathcal{I}_{0,0}$ to determine if $\mathcal{C}_{0,0}$ satisfies the λ-proximal-contraction property. The same procedure is repeated for the index sets $\mathcal{I}_{0,N-2}$, $\mathcal{I}_{N-2,0}$, and $\mathcal{I}_{N-2,N-2}$.

The λ-contractive sets turn out to satisfy the λ-proximal-contraction property for the example, so Corollary 3 applies. The admissible set is thus the intersection of these four λ-proximal-contraction sets. Numerically, the set can be represented as $\mathcal{S} = \{x : \; |x_1 - x_2| \le 0.5, \; |x_2| \le 1.79\}$. The set is depicted in Figure 6. We remark that that the same set is returned by the algorithm presented in [7].

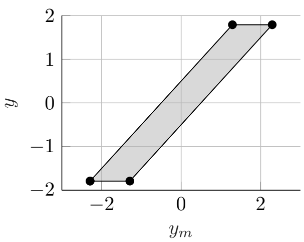

Fig. 6. Admissible set for the numerical example where the reference is treated as unobservable.

B. Observable reference signal

A reference signal for a control system is typically not unobservable. Let use therefore examine how an observable reference can be taken into consideration when calculating the admissible set and how the results may change.

Equation (6) is a model of the reference signal. It can alternatively be formulated as

$$\begin{aligned} &\forall k \in \mathbb{Z}, \quad r(k+1) = r(k) + r'(k), \\ &r(k) \in [-1, 1], \quad r'(k) \in [-0.1, 0.1], \end{aligned} \tag{7}$$

where r' is then treated as an exogenous disturbance.

It is not straightforward to fit model (7) into a framework for calculating invariant sets. The variable r cannot be bounded by including it in the constraint set, since the marginally stable model implies that the bound is reachable by the disturbance. In addition, invariant-set algorithms can only return nonempty invariant sets if the system is asymptotically stable (when disturbances are neglected). Hence, the simple model (7) is insufficient.

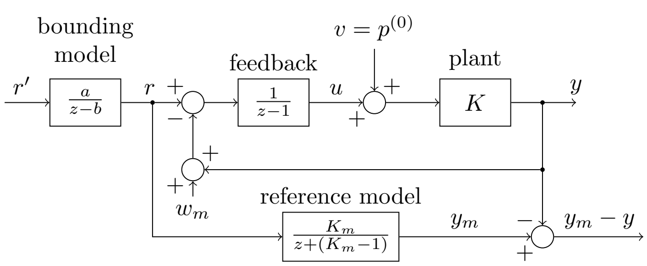

Fig. 7. System when bounding model is used to represent the reference signal. Compare with Figure 5.

To account for the simple model while also having a chance to find invariant sets, we apply a model where noise goes through an asymptotically stable filter, such as in [10]. With a first-order filter this becomes $r(k+1) = ar(k) + br'(k)$, $r'(k) \in [-1, 1]$. Figure 7 depicts the reference model in a modified block diagram. The parameters a and b need to be chosen appropriately to account for the simple model. A natural restriction is to choose a and b such that the filter is asymptotically stable and such that the maximum rate of change equals its bound when the reference signal equals its bound. With the numerical bounds for this example this leads to $|a| < 1$, $b = 0.1 + (1 - |a|)$. Hence, a can be seen as a design variable for the filter that trades off between the maximum rate of change and the maximum value for the state variable. Figure 8 depicts two choices for a.

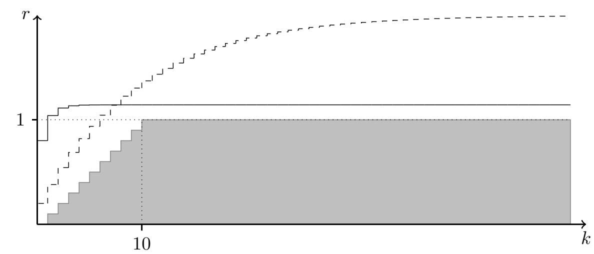

Fig. 8. A one-sided interpretation of modeling observable slowly-varying disturbances as first-order filtered bounded noise. The gray area corresponds to the simple model (7), while the two step responses illustrate two choices for a that takes the simple model into account; $a = 0.3$ (solid) and $a = 0.9$ (dashed).

Writing the modified system on the form (4), now with $x_1 = y_m$, $x_2 = y$, and $x_3 = r$ yields

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ r(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1-K_m & 0 & K_m \\ 0 & 1-K & K \\ 0 & 0 & a \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \\ r(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ -K & K & 0 \\ 0 & 0 & b \end{bmatrix}}_{B} w(k) + \underbrace{\begin{bmatrix} 0 \\ K-1 \\ 0 \end{bmatrix}}_{D} p^{(0)},$$

where we consider the two choices $a = 0.3$ and $a = 0.9$ in the calculations that follow (for comparison), and where $w = \begin{bmatrix} w_m & \tilde{v} & r' \end{bmatrix}^T$, $\tilde{v}(k) = v(k) - v(k-1)$. The disturbance set becomes $\mathcal{W} = \{w : \; |w_1| \le 0.01, \; |w_2| \le 0.1, \; |w_3| \le 1\}$, and the constraint set is $\mathcal{X} = \{x : \; |x_1 - x_2| \le 0.5\}$. No invariant sets exist for this system.

By knowledge of the bounded rates of change, we select $p^{(0)} = v$ and divide the interval into 19 subintervals, each of width 0.2. Interval $\mathcal{P}^i$ corresponds to $v \in [-1+0.1i, -0.8+0.1i]$ for $i = 0, \ldots, 19$. To apply Corollary 3, we need to be careful with our choice of λ. Due to the limited contraction for the reference model it is not possible to choose $\lambda = 0.5$ this time. Instead we choose $\lambda = 0.9$ and find the MRPI set for the system $x(k+1) = \lambda^{-1}\big(Ax(k) + Bw(k) + Dp(k)\big)$, $p \in \mathcal{P}^i$, for $i = \mathcal{I}_0 = \{0, 1\}$. This returns the sets $\mathcal{C}_0$ and $\mathcal{C}_1$. For their intersection we then evaluate if $\mathbf{Pre}\left(\bigcap_{i \in \mathcal{I}_0} \mathcal{C}_i\right) \in \mathcal{C}_i$, $i \in \mathcal{I}_0$, to determine if $\mathcal{C}_0$ satisfies the λ-proximal-contraction property. The same procedure is repeated for the index set $\mathcal{I}_{N-2}$.

It turns out that the λ-contractive sets do *not* satisfy the λ-proximal-contraction property with $a = 0.3$ while they do for $a = 0.9$. The reason for why $a = 0.3$ does not work is because the rate of change for that model is too high (see Figure 8). So Corollary 2 only applies for $a = 0.9$. With $a = 0.9$, the admissible set is the intersection of these sets and the set that describes the original knowledge that $r \in [-1, 1]$ from the simple model (7). Numerically, the set can be represented as

$$\mathcal{S} = \{x : \ |x_1 - x_2| \leq 0.5, \ |x_2 - r| \leq 2.79 \ \ |r| \leq 1\}.$$

The set is depicted in Figure 9. Note by the lower-left plot that the intersections of sets for constant references when projected onto the x_1–x_2 space becomes the set in Figure 6.

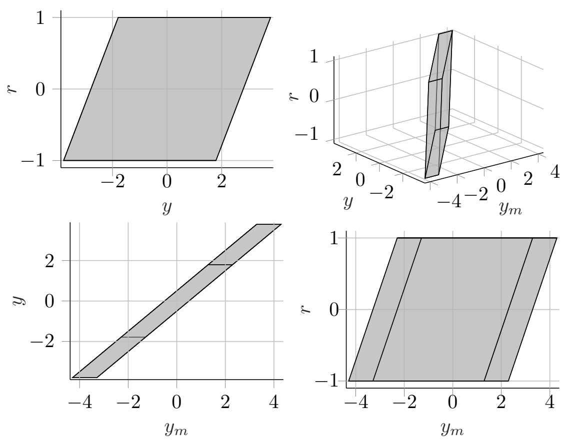

Fig. 9. Admissible set (from four different points of view) for the numerical example where the reference is treated as observable and modeled as a first-order filtered disturbance. Note the similarity with Figure 6; the lines in the lower-left plot at $x_2 = \pm 1.79$ correspond to intersections of slices with the reference held constant. This is in essence what is done by the first method that finds the set in Figure 6.

C. Discussion

While unobservable disturbances justify the method, the above example illustrates how also slowly-varying *observable* disturbances can be dealt with in at least two ways for the calculation of admissible sets. The reference is modeled as a slowly-varying parameter in the first example, while it is modeled as a state variable in the second example by augmenting the system with a bounding model. Both methods lead to similar results for the considered system.

We argue, however, that the first method is applicable for more general problems than the second. Consider for example an additional safety-constraint of $|y| \leq 2$ for the system. The first method has no problem to return the same admissible set after this constraint is added, while the second method fails to find an admissible set.

V. Summary and conclusions

Motivated by safety-critical control applications with slowly-varying parameters, the authors have in [7] developed a method to find admissible sets of initial conditions. The initial conditions are admissible in the sense that trajectories starting in the set are guaranteed to never leave the constraint set.

This contribution focuses on systems with slowly-varying additive disturbances, which is a subclass of the system class considered in [7]. By dividing the range of the slowly-varying parameter into equidistant subintervals, the λ-proximal-contraction property was introduced to derive conditions for admissible sets as intersections of λ-contractive sets that satisfy the property. A linear relation between distances to hyperplanes that define the k-step robust backward-reachable set was examined to realize that one only needs to find sets with this property at the vertices of the set of slowly-varying disturbances.

The method was derived for the scalar case and then extended to linear polytopic systems and the multidimensional case without significant effort. For a particular choice of λ, there is no *a-priori* guarantee that the method will find a set, but problem-specific intuition can help to make qualified guesses.

While the examples in Section IV were inspired by real-life safety-critical control problems, they are here adapted to make them easy to follow and to highlight the outlined method to find admissible sets. The method is most justified for unobservable disturbances, but the examples also show how observable slowly-varying disturbances can either be treated as unobservable or modeled as filtered noise, at the gain of removing slowly-varying disturbances but at the expense of introducing additional state variables. The calculations were carried out with Matlab using standard algorithms implemented by the authors.

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